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Unsteady transonic flows with shock waves in two-dimensional channels

By T. C. ADAMSON AND G. K. RICHEY
Department of Aerospace Engineering, The University of Michigan

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A two-dimensional unsteady transonic flow of a perfect gas with constant specific heats is considered, solutions being found in the form of perturbations from a uniform, sonic, isentropic flow. Longitudinal viscous stress terms are retained so that shock waves can be included. The case where the characteristic time of a temporal flow disturbance is large compared with the time taken by a sonic disturbance to traverse the transonic regime is studied. A similarity solution involving an arbitrary function of time is employed, such that the channel walls are in general not stationary. Solutions are presented for thick (shock fills transonic region) and thin (shock tends to a discontinuity) shock waves for both decelerating and accelerating channel flows. For the thin-shock case, both numerical and asymptotic solutions are given. Flow pictures illustrating variations in shock position and structure as well as velocity distributions are shown for exponentially decreasing and for harmonic temporal flow disturbances.

1. Introduction

The study of unsteady transonic channel flows with shock waves has applications in many technically important flow fields. For example, transonic flows with shocks occur under typical operating conditions in the channels between the blades in the stators and rotors in both the compressors (decelerating flow) and turbines (accelerating flow) of modern aircraft jet engines. Unsteadiness in the flow can arise from various causes such as non-uniform air inlet distributions, gusts, changes in the power setting, twisting and flexure of the blades, etc.

Although solutions for unsteady transonic flows with shocks cannot be found in the literature, several steady-flow solutions have been presented. Notable among these, from the viewpoint of the present work, are those given by Kopystynski & Szaniawski (1965), Sichel (1966) and Ryzhov (1968). Kopystynski & Szaniawski, using a series expansion, found an apparently quasi-one-dimensional solution which is of considerable interest since both accelerating and decelerating flows can be considered. Sichel and Ryzhov found that inviscid similarity transformations worked also in viscous flows and were able to find two-dimensional solutions involving shocks with thickness of the order of the axial extent of the transonic region. It is the latter approach which is used in the present study, in that similarity transformations, extended to include unsteady effects, are employed.

The present work is an extension of work done previously on inviscid unsteady transonic channel flows (Adamson 1972*a*). It will be seen that the shock solutions essentially join two of the inviscid solutions found previously, and that the various characteristic time regimes of the disturbance shown previously hold also when shocks are considered.

In the following analyses, the flow is assumed to be two-dimensional, compressible and transonic; the gas is assumed to follow the perfect-gas law and to have constant specific heats. Upstream of the transonic region, the flow is taken to be irrotational.

2. Derivation of equations

The problem considered is that of perturbations from a uniform, sonic, two-dimensional stream flowing in the X direction. If the channel half-width at the minimum area and the undisturbed flow velocity are denoted by $\bar{L}$ and $\bar{a}^*$ respectively (the overbar denotes a dimensional quantity), then X and Y are coordinate distances made dimensionless with respect to $\bar{L}$, and T is the time made dimensionless with respect to $\bar{L}/\bar{a}^*$. The independent variables are stretched as follows, with the orders of the gauge parameters to be given later:

$$X = \delta x, \quad \delta = \bar{L}_x/\bar{L}, \quad x = O(1), \tag{1a}$$

$$Y = \epsilon y, \quad \epsilon = \bar{L}_y/\bar{L}, \quad y = O(1), \tag{1b}$$

$$T = \tau t, \quad \tau = \bar{T}_{ch}/(\bar{L}/\bar{a}^*), \quad t = O(1). \tag{1c}$$

Here, $\bar{L}_x$ and $\bar{L}_y$ are lengths of the order of the physical extent of the transonic region in the X and Y directions, respectively, and $\bar{T}_{ch}$ is the characteristic time associated with the disturbance. Hence, x, y and t are of order unity in the unsteady transonic region.

Since the flow is transonic, $\delta \ll \epsilon$ (i.e. $\bar{L}_x \ll \bar{L}_y$) in general, and it is clear then that $U_{xx} \gg U_{yy}$, i.e. that longitudinal stress terms are large compared with shear stress terms. Thus, it is seen that in so far as the first-order equations are concerned, only shock waves can be considered, not boundary layers. Furthermore, since the flow is transonic, the shocks are weak, the increase in entropy being of second order within and third order across the shocks. Hence, to first order, the flow is isentropic and a velocity potential function $\Phi(X, Y, T)$, where Φ is made dimensionless with respect to $\bar{L}\bar{a}^*$, can be defined as follows:

$$\Phi(X, Y, T) = X + E_1\delta\phi(x, y, t) + \dots. \tag{2}$$

Here, $E_1 \ll 1$, $\phi_x = O(1)$ and δ is at most $O(1)$. That is,

$$\Phi_X = U = 1 + E_1\phi_x + \dots = 1 + E_1 u_1 + \dots, \tag{3a}$$

$$\Phi_Y = V = (E_1\delta/\epsilon)\phi_y + \dots = (E_1\delta/\epsilon)v_1 + \dots, \tag{3b}$$

where U and V are velocity components in the X and Y directions respectively. Thus $E_1 = O(U-1)$ is a measure of the deviation of the flow velocity from its sonic value. Finally, u_1 and v_1 are perturbation velocity components.

Expansions similar to the expansion for U, equation (3*a*), may be written for the thermodynamic variables P, ρ and $\hat{T}$ (each made dimensionless with respect to the corresponding variable in the undisturbed flow). If these expansions, (3) and (1) are substituted into the conservation equations, the usual acoustic relations between the perturbations are found, i.e.

$$P_1 = \gamma \rho_1 = [\gamma/(\gamma-1)]\hat{T}_1, \tag{4a}$$

$$P_1 = -\gamma[u_1 + (\delta/\tau)\,\phi_t]. \tag{4b}$$

The only change from the steady-flow relationships is the unsteady term in (4*b*). In deriving (4), it has been assumed that at most the shock thickness is of the same order as the extent of the transonic region in the axial direction. That is, if $\bar{L}_s$ is of the order of the thickness of a weak shock (Illingworth 1953), then

$$\frac{\bar{L}_s}{\bar{L}} = \frac{4}{Re\,E_1(\gamma+1)}\left(1+\frac{\gamma-1}{Pr}\right) = O\left(\frac{1}{Re\,E_1}\right) \leqslant \delta, \tag{5}$$

where the Reynolds and Prandtl numbers are based on the longitudinal viscosity.

The governing equation for $\phi(x,y,t)$ may be found by substituting the above expansions for the velocity components, the thermodynamic functions and the stretched variables into the general gasdynamic equation (i.e. including unsteady and viscous terms). In deriving the equation, the following relationships are employed:

$$\epsilon = 1, \quad \delta^2 = (\gamma+1)\,E_1, \tag{6a, b}$$

$$\frac{\bar{L}_s}{\bar{L}_x} = \frac{4}{Re\,E_1\delta(\gamma+1)}\left[1+\frac{\gamma-1}{Pr}\right] \equiv 4k_s, \tag{6c}$$

$$\Gamma = \frac{\gamma-1}{Pr\,[1+(\gamma-1)/Pr)]}. \tag{6d}$$

Equation (6*a*) arises since the transonic region fills the channel; the relative orders of δ and E_1 indicated by (6*b*) must hold if the governing equation is to reduce to the usual two-dimensional steady-flow equation as $\tau \to \infty$. Equation (6*c*) defines k_s as being of the order of the ratio of the thickness of a weak shock [equation (5)] to the axial extent of the transonic region. Finally, Γ is introduced for convenience. The governing equation for ϕ may then be shown to be (Adamson 1972*b*)

$$-\phi_x\phi_{xx} + \phi_{yy} - \frac{2}{\tau\delta}\phi_{xt} - \frac{1}{\tau^2}\phi_{tt} + k_s\left[\phi_{xxx} + \frac{\delta}{\tau}\Gamma\phi_{xxt}\right] = 0. \tag{7}$$

The first four terms are those derived previously for inviscid unsteady flows (Adamson 1972*a*), while the last contains the effects of viscosity through the parameter k_s. When $k_s = O(1)$, $\bar{L}_s = O(\bar{L}_x)$; for this case, hereafter referred to as the thick-shock case, the viscous terms must be retained. As k_s becomes smaller, which corresponds to the Reynolds number becoming larger, the effects of viscosity become small in the main part of the flow, being confined to a thinner and thinner shock region until $k_s = 0$, when the shock is a discontinuity with inviscid flow upstream and downstream of it. The limiting case is referred to as the thin-shock case. Equation (7) is the unsteady counterpart of the equation studied by Sichel (1966) in his analysis of steady viscous transonic flows.

The same regimes for τ in terms of δ arise in the viscous case as were found in the inviscid case (Adamson 1972*a*). They are (i) $\tau = O(\delta)$, (ii) $\tau = O(1)$ and (iii) $\tau = O(1/\delta)$. Just as in the inviscid study, the third case is chosen for study; many physical problems associated with unsteady flows in jet engines fall into this time regime. Thus, for this study

$$\tau = (k\delta)^{-1}, \tag{8}$$

where k is an arbitrary constant of order unity. Using (8), (3) and the irrotationality condition, one can write (7) entirely in terms of u_1; thus

$$k_s u_{1xxx} - (u_{1x})^2 - u_1 u_{1xx} + u_{1yy} - 2k u_{1xt} = 0. \tag{9}$$

Just as in the inviscid case (Adamson 1972*a*), it can be shown that for the chosen τ regime, equation (8), the equation for the wall may be written in stretched variables as

$$y_w = y_\alpha + (\gamma+1) E_1^2 f_w(x,t), \quad \partial f_w/\partial x = v_w, \tag{10a, b}$$

where y_α is a constant and where $(f_w)_t$ drops out of (10*b*) because for this τ regime $\tau \gg \delta$. Equation (10*b*) expresses the fact that the wall is instantaneously a streamline. f_w is a function of t as well as x since in general the wall may be moving. Finally, it can be shown that $\bar{a}^*$ is a constant, so that the sonic line is defined by $u_1 = 0$.

3. Similarity solutions

Solutions of (9) may be found by employing the same similarity transformation as was used for the inviscid problem (Adamson 1972*a*). This transformation is an extension of the one used by Tomotika & Tamada (1950) for inviscid and Sichel (1966) for viscous channel flows. It is worthwhile to point out again that similarity solutions are not general in that one cannot impose arbitrary boundary and initial conditions. In this case, however, these conditions correspond to those associated with channel flows, so the result is a solution which is very instructive but which involves a minimum of mathematical and computational difficulties. The transformation is written as follows:

$$s = x + by^2 + \beta(t), \tag{11a}$$

$$u_1 = z(s) + 4b^2y^2 - 2k\beta', \tag{11b}$$

where b is a constant, $\beta(t)$ is an arbitrary function of time and the prime on β indicates the derivative with respect to time. Using the irrotationality condition $u_{1y} = v_{1x}$, one can show that

$$v_1 = 2byz + 8b^2yx + \tfrac{8}{3}b^3y^3 + y(8b^2\beta - 4k^2\beta'') \tag{12}$$

and finally that (9) becomes

$$k_s z''' - zz'' - (z' - 4b)(z' + 2b) = 0, \tag{13}$$

where a prime denotes differentiation with respect to s. Thus, a similarity solution results for the viscous as well as the inviscid unsteady transonic channel flow; the most ineresting point of the transformation is that an arbitrary function of time $\beta(t)$ is involved. The form of β is related to the initial and boundary conditions. A random disturbance which dies out or a cyclic disturbance may be represented by an exponential or a sinusoidal form for β, respectively. For any chosen β, there is some wall motion, and it is not possible to isolate the effects of the moving wall or the variable incoming stream.

Equation (13) may be integrated once to give

$$k_s z'' - zz' + 2bz + 8b^2 s = 0, \tag{14}$$

where the constant of integration has been absorbed in s, i.e. x can be measured from any arbitrary location.

When $k_s \equiv 0$, the flow can be considered to be either completely inviscid or inviscid up to and downstream of an infinitesimally thin shock. If the former case is considered first, the solutions are those given by Tomotika & Tamada (1950) for steady flows and exploited later for unsteady flows (Adamson 1972*a*). Thus

$$(z-4bs)^2(z+2bs) = \alpha^3/4b^3, \tag{15}$$

where α is a constant of integration which characterizes the inviscid solution curves, i.e. α = constant along a given solution curve.

Equation (14) is essentially the equation studied by Sichel (1966) in his analysis of shocks in accelerating channel flows, where $k_s = O(1)$. Here we consider the unsteady counterparts of his solutions. In addition, solutions for decelerating flows with $k_s = O(1)$, both steady and unsteady, and solutions for $k_s \ll 1$ for accelerating and declerating flows will be presented.

Equation (14) has been solved both numerically and using asymptotic methods. These solutions are discussed in the following sections.

3.1. *Numerical solutions: thick shocks* ($k_s = O(1)$)

Numerical solutions to (14) are shown in figure 1. The dashed curves are inviscid solutions, equation (15), for various values of α, the subscripts u and d on α denoting upstream and downstream (of the shock) conditions respectively. Since, from (11*b*), $u_1 = z$ along the axis of a steady flow, it is seen that inviscid solutions for $\alpha > 0$ correspond to flows everywhere supersonic, through a converging or diverging channel; solutions for $\alpha < 0$ correspond to flows subsonic along the axis, but with supersonic pockets away from the axis (Taylor flow).

Curve (*a*) in figure 1 is a numerical solution of (14) for $k_s = 1$ and $b = \frac{1}{2}$ which starts close to the inviscid acclerating-flow solution $z = 4bs$ (Meyer flow) and then jumps, through a shock, to a downstream inviscid-flow solution corresponding to $\alpha_d = -2{\cdot}82$. This solution is similar to those shown by Sichel (1966) for steady flow. Upon transforming the solution to physical co-ordinates, for various functions β, corresponding unsteady flows with thick shocks can be considered. Such flow pictures will be shown later. Solution (*a*) was found by starting the solution near $z = 4bs$ with slope near $4b$. The asymptotic downstream inviscid curve was found by numerically matching the solution with inviscid solutions and calculating the α indicated.

It was reasoned that as for the accelerating flows studied by Sichel (1966) it should be possible to consider decelerating supersonic flows with shocks, and

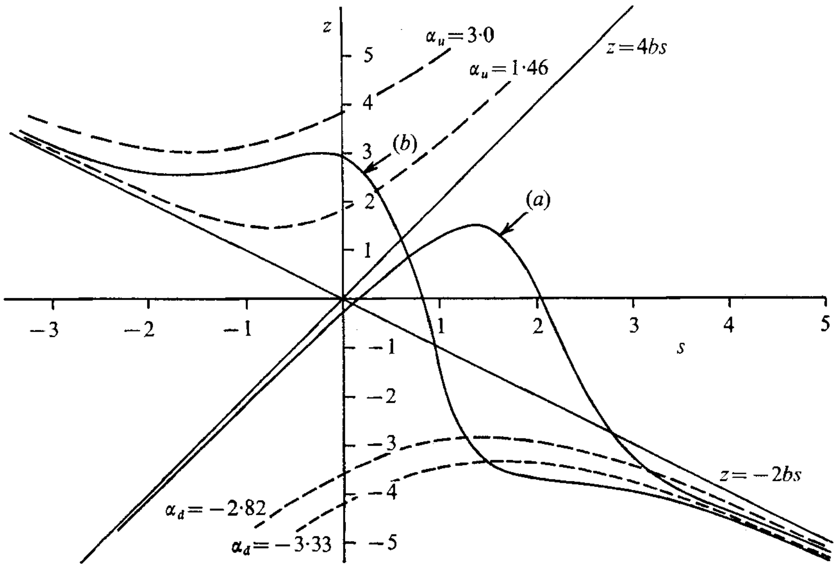


FIGURE 1. z against s, showing inviscid solutions [equation (15)] for various α (dashed curves) and solutions to the viscous equation (14) for $k_s = 1$, $b = \frac{1}{2}$. Solution (a) (shock in accelerating flow) starts asymptotic to $z = 4bs$ and ends asymptotic to the $\alpha_d = -2{\cdot}82$ inviscid solution. Solution (b) (shock in decelerating flow) starts asymptotic to $\alpha_u = 1{\cdot}46$ and ends asymptotic to $\alpha_d = -3{\cdot}33$ inviscid solutions.

such turned out to be the case. Curve (b) in figure 1 thus illustrates a flow which starts as a decelerating inviscid supersonic flow which reaches a minimum velocity, accelerates slightly and then jumps, through a shock, to a downstream inviscid-flow solution and continues to decelerate. The calculation was again made for $k_s = 1$ and $b = \frac{1}{2}$. It was started at a given point with a given slope in the supersonic region and matched numerically with downstream inviscid solutions to obtain α_d. Then backward integration was carried out from the initial point and α_u was found by comparison with inviscid solutions for $\alpha > 0$. The reason why backward integration is necessary will be discussed later.

It is clear that, in both cases illustrated in figure 1, the position of the shock depends on the values of α being considered and hence on downstream conditions in the case of simple acclerating flows and on both downstream and upstream conditions in the case of decelerating flows.

3.2. *Thin-shock solutions* ($k_s \equiv 0$)

When $k_s = O(1)$, then by (6c) and (6b), $Re = O(E_1^{-\frac{3}{2}})$. For typical values of E_1, this means that Re is at most of order 10^2–10^3, which is very small compared, say, with the Re found in typical turbomachines. Thus, it is necessary to study cases where $k_s \ll 1$. In this section, we first consider the case where k_s has gone to the limiting value of zero, so that the shock is a discontinuity imbedded in an otherwise inviscid transonic flow. In the next subsection, the case where $k_s \ll 1$, but not zero, is considered.

The case $k_s = 0$ can be studied by considering the general equations for an oblique shock which is itself moving with time. Since for the τ regime considered the first-order conservation equations do not contain any time derivatives, it is clear that the jump conditions across the shock, relative to the shock, will be the familiar steady jump conditions. If expansions (3) for the velocity components are substituted into the jump conditions and the similarity solutions (11*b*) and (12) are substituted for the velocity perturbations, it is found that the similarity solution holds across the shock and that the jump condition and shock-wave velocity U_s are respectively (Adamson 1972*b*)

$$z_d = -z_u, \tag{16a}$$

$$U_s = -E_1(2k\beta') + \dots. \tag{16b}$$

Equation (16*a*) is thus a generalized jump condition corresponding to that across a weak normal shock, where $u_d = -u_u$, u being the perturbation velocity. Finally, it is also found that the shock is along a line s = constant.

The jump condition (16*a*) and the inviscid solution (15) can be used to derive an equation for the position of the shock in terms of the upstream and downstream α values. Thus, if s_0 denotes the value of s at the shock location in the z, s plane, then it can be shown (Adamson 1972*b*) that at the shock

$$z_u = (\alpha_u^3 - \alpha_d^3)^{\frac{1}{3}}/2b = -z_d, \tag{17a}$$

$$s_0 = \begin{cases} \pm\dfrac{(\alpha_u^3-\alpha_d^3)^{\frac{2}{3}}}{4b^2}\cos\left(\dfrac{\omega}{3}+\dfrac{4\pi}{3}\right) & \text{for} \quad |\alpha_d| \lessgtr \alpha_u, \quad (17b) \\ 0 & \text{for} \quad |\alpha_d| = \alpha_u, \quad (17c) \end{cases}$$

where
$$\omega = \cos^{-1}[|\alpha_u^3 + \alpha_d^3|/(\alpha_u^3 - \alpha_d^3)]. \tag{18}$$

Equations (17) and (18) hold not only for the case of decelerating flow, where α_u and α_d both exist, but also for accelerating flow. In the latter case, $\alpha_u \equiv 0$.

The solutions given in this section hold, of course, for both steady and unsteady flow, the differences between the two arising only as the transformation is made back to the physical flow. Typical solutions in the z, s plane are shown in figure 2. Again, the dashed lines show inviscid solutions for the indicated values of α, and the solid lines show solutions including shocks. Thus, curve (*a*) shows a solution for an accelerating flow which starts along $z = 4bs$ and then jumps, through a shock, to an inviscid flow corresponding to $\alpha_d = -0{\cdot}10$. Curve (*b*) is the solution for a decelerating supersonic flow which first decelerates, then begins to accelerate along an $\alpha_u = 0{\cdot}2$ inviscid-flow solution and then jumps, through a shock, to an $\alpha_d = -0{\cdot}40$ inviscid curve and continues to decelerate.

3.3. *Thin-shock solutions* ($k_s \ll 1$)

Here, we consider the case where the shock thickness is small compared with the axial extent of the transonic region, but not negligible; $k_s \ll 1$. In this case, referring to (14), the problem is a classical singular perturbation problem, and the method of matched asymptotic expansions (Van Dyke 1964; Cole 1968) can be used. This approach is more desirable than the previously mentioned method of obtaining numerical solutions because the problem is in general a two-point boundary-value problem. That is, one can in general choose both the inviscid

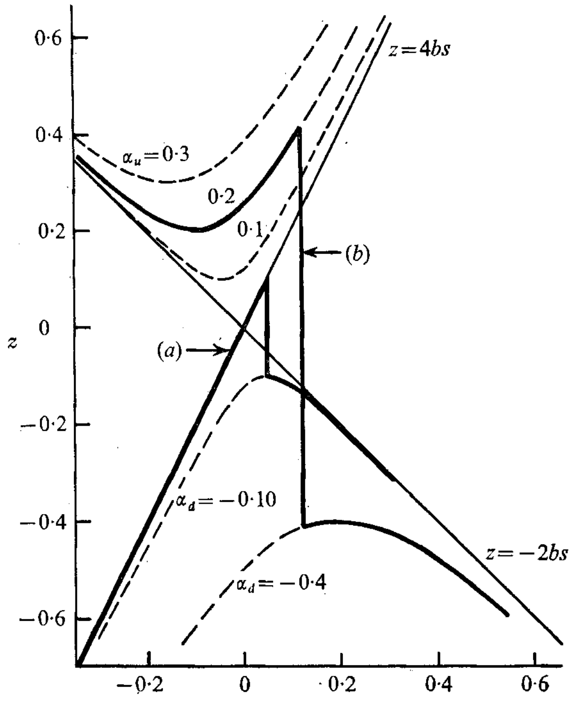


FIGURE 2. z against s, showing inviscid solutions [equation (15)] for various α (dashed curves) and solutions for flows with shocks in the limit $k_s = 0$ (shock infinitesimally thin). Solution (*a*) is a shock in accelerating flow; solution (*b*), a shock in decelerating flow.

solutions to which the solution is asymptotic upstream and downstream of the shock. In so far as numerical solutions are concerned, this means a trial-and-error procedure involving various starting values. On the other hand, approximate analytical expansions allow one to form an upstream asymptotic expansion, the constants of which are written in terms of α_u and α_d. This asymptotic expansion may then be used to start a numerical solution which will start and finish asymptotic to the desired inviscid curves. Alternatively, the solutions found may be used throughout the whole flow region, if approximate solutions valid to a given order of approximation are acceptable in the problem at hand.

In the following the flow is divided into three regions: the inner shock region and the two outer regions upstream and downstream of the shock wave. The solutions found are asymptotic expansions, valid in the limit as $k_s \rightarrow 0$. In the interests of brevity, and because the analysis is relatively straightforward, only a brief outline of the analysis is given here. Details may be found in Adamson (1972*b*).

Inner region. If, again, s_0 is defined as the value of s at which the shock exists in the limit as $k_s \rightarrow 0$, then following Sichel (1971) s is stretched and z is expanded as follows in the inner shock region:

$$\tilde{s} = (s - s_0)/k_s, \tag{19a}$$

$$z(s) = g(\tilde{s}) = g_0(\tilde{s}) + k_s g_1(\tilde{s}) + \dots. \tag{19b}$$

The governing equations for g_0 and g_1, found by substituting (19) into (14), are

$$g_0'' - g_0 g_0' = 0, \tag{20a}$$

$$g_1'' - (g_1 g_0)' + 2bg_0 + 8b^2 s_0 = 0, \tag{20b}$$

where a prime indicates differentiation with respect to $\tilde{s}$. Referring to (20), it is seen that the solution in the inner region is simply a perturbation from the Taylor (1910) weak-shock solution. Thus, solving (20) and substituting the results into (19b), one finds for $g(\tilde{s})$

$$\begin{aligned} g = &-\tilde{c}_1 \tanh \tilde{r} + k_s(\cosh \tilde{r})^{-2}\Bigg[\frac{4b}{\tilde{c}_1}\Bigg\{\ln \cosh \tilde{r}\,(\sinh \tilde{r} \cosh \tilde{r} + \tilde{r}) \\ &- \tfrac{1}{2}(\sinh \tilde{r} \cosh \tilde{r} - \tilde{r}) - \int_0^{\tilde{r}} \tilde{r}_1 \tanh \tilde{r}_1\, d\tilde{r}_1\Bigg\} \\ &- \frac{16b^2 s_0}{\tilde{c}_1^2}\left\{\tilde{r} \sinh \tilde{r} \cosh \tilde{r} - \frac{\sinh^2 \tilde{r}}{2} + \frac{\tilde{r}^2}{2}\right\} + \frac{\tilde{c}_3}{2}\{\sinh \tilde{r} \cosh \tilde{r} + \tilde{r}\} + g_1(0)\Bigg] + \ldots, \end{aligned} \tag{21a}$$

$$\tilde{r} = \tfrac{1}{2}\tilde{c}_1 \tilde{s} + \tilde{c}_2. \tag{21b}$$

It is seen that, in the limit as $k_s \to 0$, equation (16a) is recovered, as should be the case. The asymptotic expansions for $g(\tilde{s})$ needed for matching are, as $\tilde{s} \to \pm\infty$ (i.e. $\tilde{r} \to \pm\infty$),

$$\begin{aligned} g = \mp \tilde{c}_1[1 - 2\exp(\mp 2\tilde{r}) + \ldots] + k_s\Bigg[\tilde{r}\left(\frac{4b}{\tilde{c}_1} \mp \frac{16b^2 s_0}{\tilde{c}_1^2}\right) \mp \frac{4b}{\tilde{c}_1}(\ln 2 + \tfrac{1}{2}) \\ + \frac{8b^2 s_0}{\tilde{c}_1^2} \pm \frac{\tilde{c}_3}{2} \pm \frac{4}{\tilde{c}_1}\left(2b + \frac{8b^2 s_0}{\tilde{c}_1}\right)\tilde{r}^2 \exp(\mp 2\tilde{r}) + \ldots\Bigg] + \ldots, \end{aligned} \tag{22}$$

where, in each case, the upper sign holds for the limit as $\tilde{s} \to +\infty$ and the lower as $\tilde{s} \to -\infty$, and where an exponentially small term is retained in g_1 for numerical work to be considered later. As may be seen in (22), $g_1(0)$ does not appear and thus is not obtained from matching to this order, just as $\tilde{c}_2$ (in g_0) is not found by matching to zeroth order. This simply means that the shock location (e.g. the point at which $g = 0$) is known to the same order of approximation as the solutions.

Outer region. The outer regions are handled in general by assuming a solution which is asymptotic to an inviscid solution. As will be seen, this covers all cases for which $|\alpha| > 0$, but is not valid for the single important case $\alpha = 0$, where the inviscid solution is $z_i = 4bs$ (Meyer flow).

In general the solution is of the form

$$z = z_i(s) + k_s z_1(s) + \ldots. \tag{23}$$

If (23) is substituted into (14), the governing equations for z_i and z_1 are found to be

$$-z_i z_i' + 2bz_i + 8b^2 s = 0, \tag{24a}$$

$$-(z_1 z_i)' + 2bz_1 + z_i'' = 0. \tag{24b}$$

Thus, z_i is the solution to the inviscid equation (equation (14) with $k_s = 0$), and solutions for z_i are given by (15), with α = constant. The solution for z_1 is given by

$$z_i z_1 \left(\frac{4b - z_i'}{2b + z_i'}\right)^{\frac{1}{3}} - \left[z_i z_1\left(\frac{4b - z_i'}{2b + z_i'}\right)\right]_{s_0}^{\frac{1}{3}} = \int_{s_0}^{s} z_i'' \left(\frac{4b - z_i'}{2b + z_i'}\right)^{\frac{1}{3}} ds_1, \tag{25}$$

$-z_i'(s_0)/2b$	$I(s_0)$
1·00	0
0·50	1·331523
0·00	2·059548
−0·50	2·620750
−1·00	3·068191
−1·50	3·414208
−2·00	3·625950

TABLE 1. Value of integral $I(s_0)$ for $b = \frac{1}{2}$

where the integration is carried out from s_0 to s to facilitate expansions for s near s_0. It can be seen from (25) that, as was mentioned above, this general solution is not valid for $z_i = 4bs$.

In order to match solutions, it is necessary to find expansions for z_i and z_1 both for s near s_0 and for $|s| \gg 1$. These expansions are as follows.

For $s - s_0 \ll 1$, the expansion for z, written now in terms of the inner variable $\tilde{s}$ for ease of matching, is

$$z = A + k_s(B\tilde{s} + z_1(s_0)) + O(k_s^2), \tag{26}$$

where

$$A = z_i(s_0), \tag{27a}$$

$$B = z_i'(s_0) = 2b + 8b^2 s_0/A. \tag{27b}$$

Equation (26) is the general solution which may be specialized by using A_u, B_u and $[z_i(s_0)]_u$ (upstream values) or A_d, B_d and $[z_i(s_0)]_d$ (downstream values).

For $|s| \gg 1$, the expansion for z is

$$z = -2bs + \frac{\alpha^3}{144b^5 s^2} + \ldots + k_s \frac{\alpha}{12 \times 2^{\frac{1}{3}} b^3 s^2} \times \left[z_1(s_0) A \left(\frac{4b - B}{2b + B}\right)^{\frac{1}{3}} - 2bI(s_0) + \frac{\alpha^2 2^{\frac{1}{3}}}{8b^3 s^2} + \ldots \right] + \ldots, \tag{28}$$

where

$$I(s_0) = \int_{-B/2b}^{1} \left(\frac{2+\zeta}{1-\zeta}\right)^{\frac{1}{3}} d\zeta. \tag{29}$$

Typical values for $I(s_0)$ as a function of the lower limit are given in table 1. Again, (28) is the general solution; specific solutions upstream ($s \to -\infty$) or downstream ($s \to \infty$) of the shock are obtained by using the proper upstream or downstream values of A, B, $z_1(s_0)$, α and $I(s_0)$.

It is seen in (28) that both the second term of the inviscid expansion and the first-order term due to viscous effects tend to zero as s^{-2}. Thus, it would be possible to consider a case where the viscous solution would not tend to this inviscid solution before approaching $-2bs$. However, the problem considered is that in which solutions tend to the inviscid solutions (α = constant) both upstream and downstream of the shock; for example, when $k_s = 0$, only inviscid solutions upstream and downstream of the shock are involved. Hence, the boundary conditions here are that the solutions become asymptotic to inviscid solutions before they become asymptotic to $z = -2bs$, for any α. Thus, both upstream and downstream of the shock,

$$z_1(s_0) = \frac{2b}{A}\left(\frac{2b+B}{4b-B}\right)^{\frac{1}{3}} I(s_0), \tag{30}$$

where specific upstream or downstream values are used for A, B and $I(s_0)$. With the upstream and downstream values of $z_1(s_0)$ known, the solution (25) for z_1 may be written in terms of known quantities; in particular, all constants in the expansion (26) near $s = s_0$ are known before matching. Matching serves to locate the shock and set the parameters associated with the shock structure (inner solution).

Before matching, it is necessary to consider the outer solution not covered by the preceding analysis. As was illustrated previously, this singular behaviour occurs when the inviscid upstream flow is the so-called Meyer flow where $z_i = 2bs$. Physically, the reason for this occurrence is the fact that, because the inviscid-flow solution has a constant acceleration, $u_{xx} = z_i'' = 0$; even if longitudinal viscosity is accounted for, no viscous effects from the basic solution are seen ahead of the shock. As a result, limit expansion techniques simply do not give valid solutions; perturbations from z_i involve exponentially rather than algebraically small terms. Since terms of order k_s are being considered, no further computations are really necessary as far as the asymptotic solutions are concerned. However, in order to start a numerical solution upstream of the shock, an asymptotic expansion is needed because, in the region where z is slightly less than $4bs$, backward integration from the shock is unstable. A solution may be found using a two-variable approach. Details are given in Adamson (1972*b*). The asymptotic forms for z necessary for matching with the inner solution (in terms of $\tilde{s}$) and for s large and negative are found to be, respectively,

$$z = 4bs_0 + k_s 4b\tilde{s} + \hat{\epsilon}_0(k_s)\, k_s^{\frac{1}{4}} \exp(2bs_0^2/k_s)\, \hat{c}_2(2/\pi)^{\frac{1}{2}} \exp(4bs_0\tilde{s})\,(2b^{\frac{1}{2}}s_0)^{-\frac{1}{2}} + \ldots, \tag{31a}$$

$$z = 4bs - \exp(-2bs_0^2/k_s)\, 8b^2 s_0^{\frac{3}{2}}(-2s)^{-\frac{1}{2}} \exp[-(1+2^{-\frac{4}{3}}I(s_0))] \times [1 - 3k_s/32b(-s)^2 + \ldots], \tag{31b}$$

where $\hat{\epsilon}_0(k_s)$ and $\hat{c}_2$ are parameters found later by matching.

Matching. In this subsection matching between the inner (shock) and outer (inviscid plus small viscous effects) solutions is demonstrated for two problems. The first is the so-called decelerating-flow problem where the flow upstream of the shock is everywhere supersonic, corresponding to an inviscid-flow solution with $\alpha = \alpha_u > 0$, and the flow downstream is subsonic, corresponding to an inviscid-flow solution with $\alpha = \alpha_d < 0$. The second is the so-called accelerating-flow problem where the flow upstream of the shock is a simple Meyer flow ($\alpha = 0$, $z_i = 4bs$) and the flow downstream of the shock is subsonic ($\alpha = \alpha_d < 0$). Reference is made to figure 1, where comparable solutions, but for $k_s = 1$, are shown.

For the decelerating-flow problem, comparison of (22) and (26) both upstream ($\tilde{s} \to -\infty$) and downstream ($\tilde{s} \to \infty$) of the shock leads to the following relations, where s_0 is given by (17):

$$\tilde{c}_1 = A_u = -A_d, \tag{32a}$$

$$(2b + [8b^2 s_0/A])_{u,d} = B_{u,d}, \tag{32b}$$

$$\tilde{c}_2 = \frac{A_u}{2(B_d+B_u)}\left\{[z_1(s_0)]_d + [z_1(s_0)]_u - \frac{16b^2 s_0}{A_u^2}\right\}, \tag{32c}$$

$$\tilde{c}_3 = \frac{2}{B_d+B_u}\left\{B_u[z_1(s_0)]_d - B_d[z_1(s_0)]_u + \frac{4b}{A_u}(\ln 2 + \tfrac{1}{2})(B_u+B_d) - \frac{8b^2 s_0}{A_u^2}(B_u - B_d)\right\}. \tag{32d}$$

For the acclerating-flow problem, (22) is matched with (31a) upstream of the shock ($\tilde{s} \rightarrow -\infty$) and with (26) downstream of the shock ($\tilde{s} \rightarrow \infty$). The results are

$$\tilde{c}_1 = 4bs_0 = A_u = -A_d, \tag{33a}$$

$$\hat{\epsilon}_0(k_s)\, k_s^{\frac{1}{4}} = \exp(-2bs_0^2/k_s), \tag{33b}$$

$$\hat{c}_2 = -8\pi^{\frac{1}{2}} b^{\frac{5}{4}} s_0^{\frac{3}{2}} \exp(2\tilde{c}_2), \tag{33c}$$

$$\tilde{c}_2 = -\tfrac{1}{2}[2^{-\frac{4}{3}} I(s_0) + 1], \tag{33d}$$

$$\tfrac{1}{2}\tilde{c}_3 = \ln 2/s_0 + [z_1(s_0)]_d. \tag{33e}$$

Thus, one can find all the constants for the solutions in the inner and outer regions for either problem and form composite expansions, valid for the entire region, if desired. However, it is simpler to integrate the original differential equation (14) numerically, using asymptotic solutions to compute starting conditions for desired initial and final inviscid curves.

4. Numerical computations for $z(s)$ ($k_s \ll 1$)

It is known from the previous section that the outer upstream solution is independent of any shock parameters, in the case of decelerating flow. Hence, if one is to find an asymptotic form to start a numerical solution exhibiting the desired shock conditions, the inner solution must be employed in the overlap region, where both inner and outer solutions are valid. Moreoever, the starting-point must be such that numerically the asymptotic form is still valid, but on the other hand, the exponential function which gives the first effects of the shock is not only non-negligible, but also larger than the terms neglected. Using (22) (for $\tilde{s} \rightarrow -\infty$) and (32), these conditions may be stated as follows:

$$2A_u \exp(A_u\tilde{s} + 2\tilde{c}_2) = Fk_s(B_u\tilde{s} + [z_1(s_0)]_u), \tag{34a}$$

$$(A_u\tilde{s} + 2\tilde{c}_2)^2 \ll 2A_u^2/Fk_sB_u, \tag{34b}$$

where $F < 1$ is an arbitrary factor. For proper values of F, the solution is insensitive to small variations in F; in the examples to be shown, $F = 0{\cdot}1$ proved to be such a value. Equations (34) are used to find a starting value for $\tilde{s}$ and thus s. Then, corresponding values for z and z' are found using (22). Numerical integration is carried out both forwards and backwards from the starting-point. Such a solution cannot be obtained by starting forward integration at a large and negative s, because of the obvious local instability of (14), as evidenced by the shock solutions; such shock-like behaviour can be caused by any small error. On the other hand, backward integration upstream of the shock is stable.

Typical solutions for $z(s)$ are shown in figure 3 for $k_s = 10^{-2}$ and $k_s = 10^{-3}$, with $b = \frac{1}{2}$. Since $u_1 = z$ along the axis of a steady channel flow, these solutions

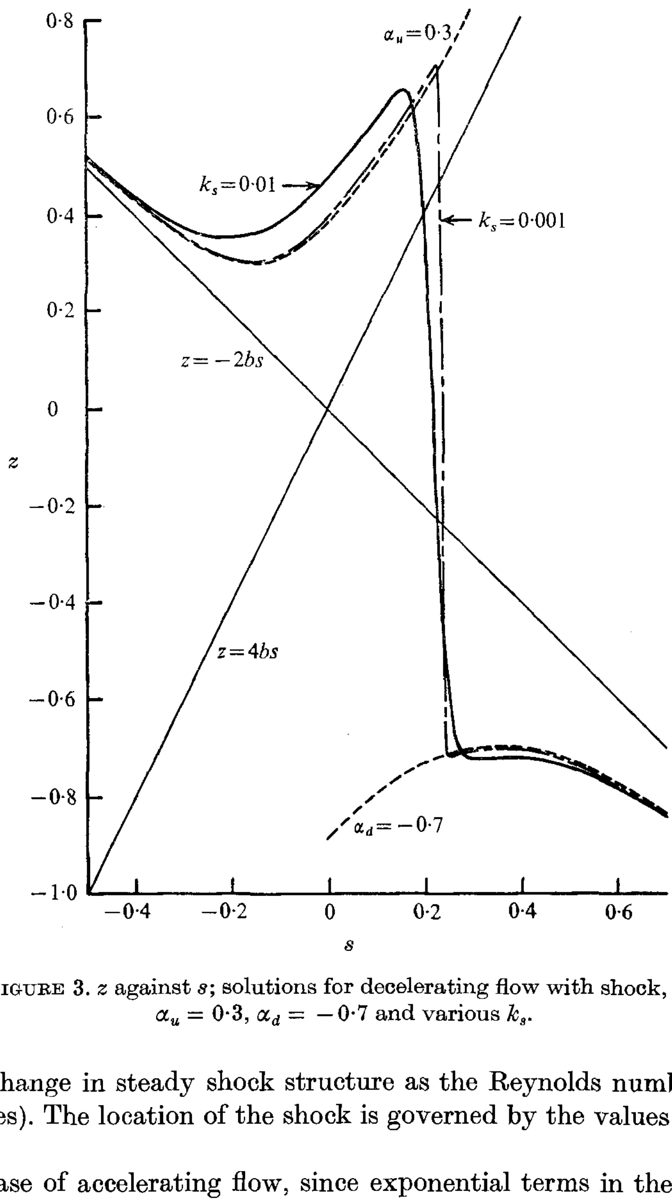


FIGURE 3. z against s; solutions for decelerating flow with shock, for $\alpha_u = 0{\cdot}3$, $\alpha_d = -0{\cdot}7$ and various k_s.

show the change in steady shock structure as the Reynolds number increases (k_s decreases). The location of the shock is governed by the values of α_u and α_d chosen.

In the case of accelerating flow, since exponential terms in the inner shock solution are matched with corresponding terms in the outer upstream solution, information concerning the shock location is carried in the outer upstream solution, and it may be used to start solutions at any point desired. For the calculations shown here, the asymptotic solution for s large and negative [equation (31b)] was chosen for computation of starting values. Typical calculations for $z(s)$ are shown in figure 4, where again $b = \frac{1}{2}$ and $k_s = 10^{-2}$ and 10^{-3}. Thus, the solutions in this form show the change in steady-flow shock structure along the centre-line of a channel flow, as the Reynolds number increases. The large circles show points calculated using the asymptotic expansions; the agreement with the numerical solutions is seen to be quite good.

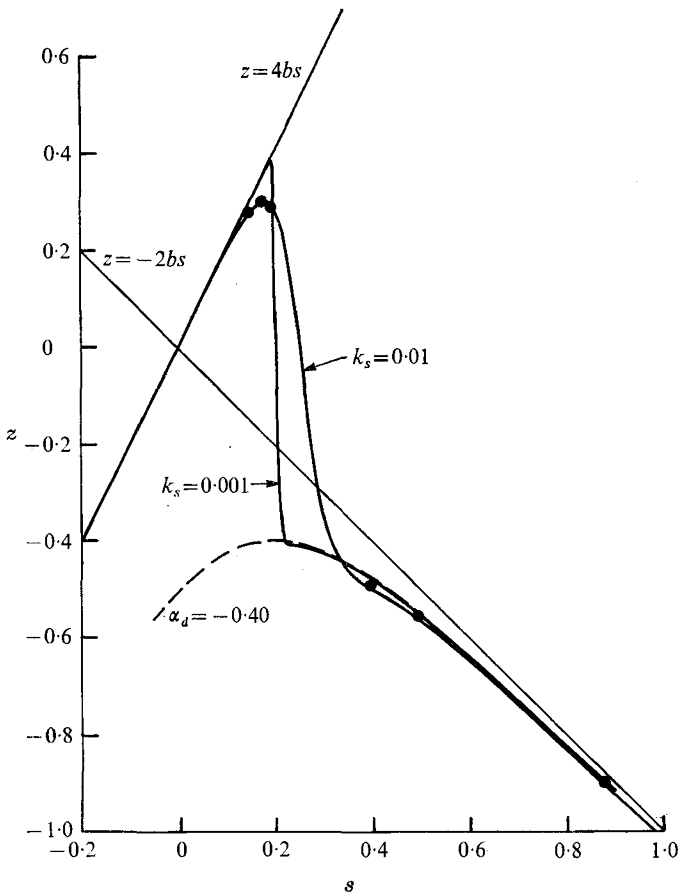


FIGURE 4. z against s; solutions for accelerating flow with shock, for $\alpha_d = -0{\cdot}4$ and various k_s. ●, points calculated using asymptotic expansions for $k_s = 0{\cdot}01$ [equations (31) and (23)].

5. Flow-field calculations

The first step in the flow-field calculations is to obtain $z(s)$. Then, the perturbation velocity components can be obtained as functions of x, y and t from (11) and (12). Finally, the wall and streamline co-ordinates may be obtained by combining the integral of (10b) with (10a). A general integral of (10b) can be found by using (14) to replace $2bz$ in v_w [see equation (12)]. If this result is substituted for f_w in (10a), then the equation which holds for the wall or streamline is, in stretched co-ordinates,

$$y_{SL} = y_\alpha \left[1 + (\gamma+1) E_1^2 \left\{ -k_s(z' - z'_\alpha) + \frac{(z^2 - z_\alpha^2)}{2} - (\tfrac{16}{3} b^2 y_\alpha^2 + 4k^2 \beta'')(x - x_\alpha)\right\}\right], \quad (35)$$

where the subscript α here refers to the initial point of the calculation. In this calculation, then, since y varies from y_α by only a small quantity,

$$s_\alpha = x_\alpha + b y_\alpha^2 + \beta(t), \quad s = x + b y_\alpha^2 + \beta(t), \quad z_\alpha = z(s_\alpha). \qquad (36a,b,c)$$

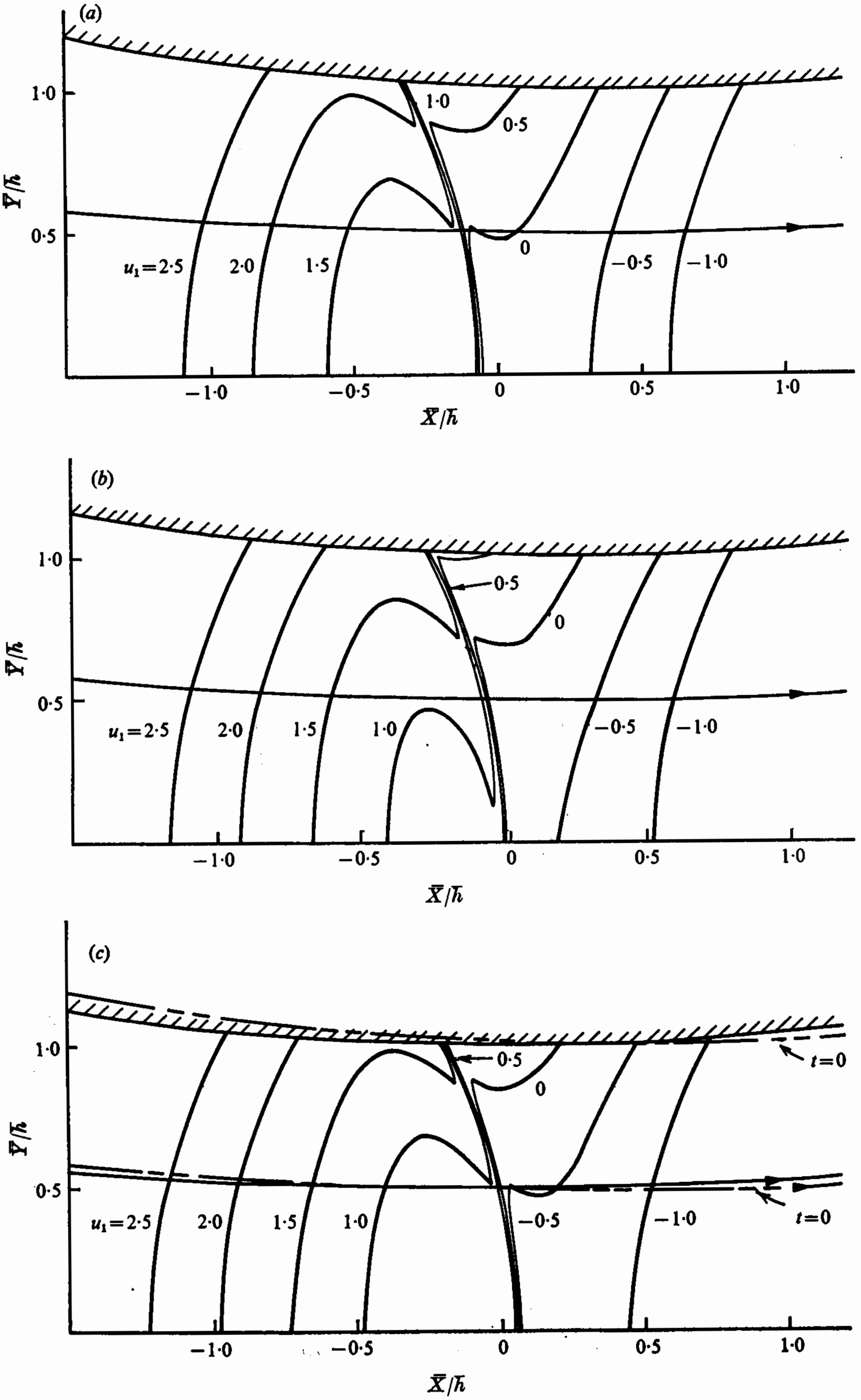


FIGURE 5. Flow pictures for thin shock in unsteady decelerating flow. $k_s = 0{\cdot}01$, $\alpha_u = 0{\cdot}50$, $\alpha_d = -0{\cdot}70$, $b = \frac{1}{2}$, $k = 1$, $\gamma = 1{\cdot}4$, $E_1 = 0{\cdot}1$, $\beta = \frac{1}{4}e^{-2t}$ (steady-state flow as $t \to \infty$, see (c).) —→—, streamline; ——, isotach. (a) $t = 0$, (b) $t = 0{\cdot}35$, (c) $t = \infty$, showing overall wall and streamline variation.

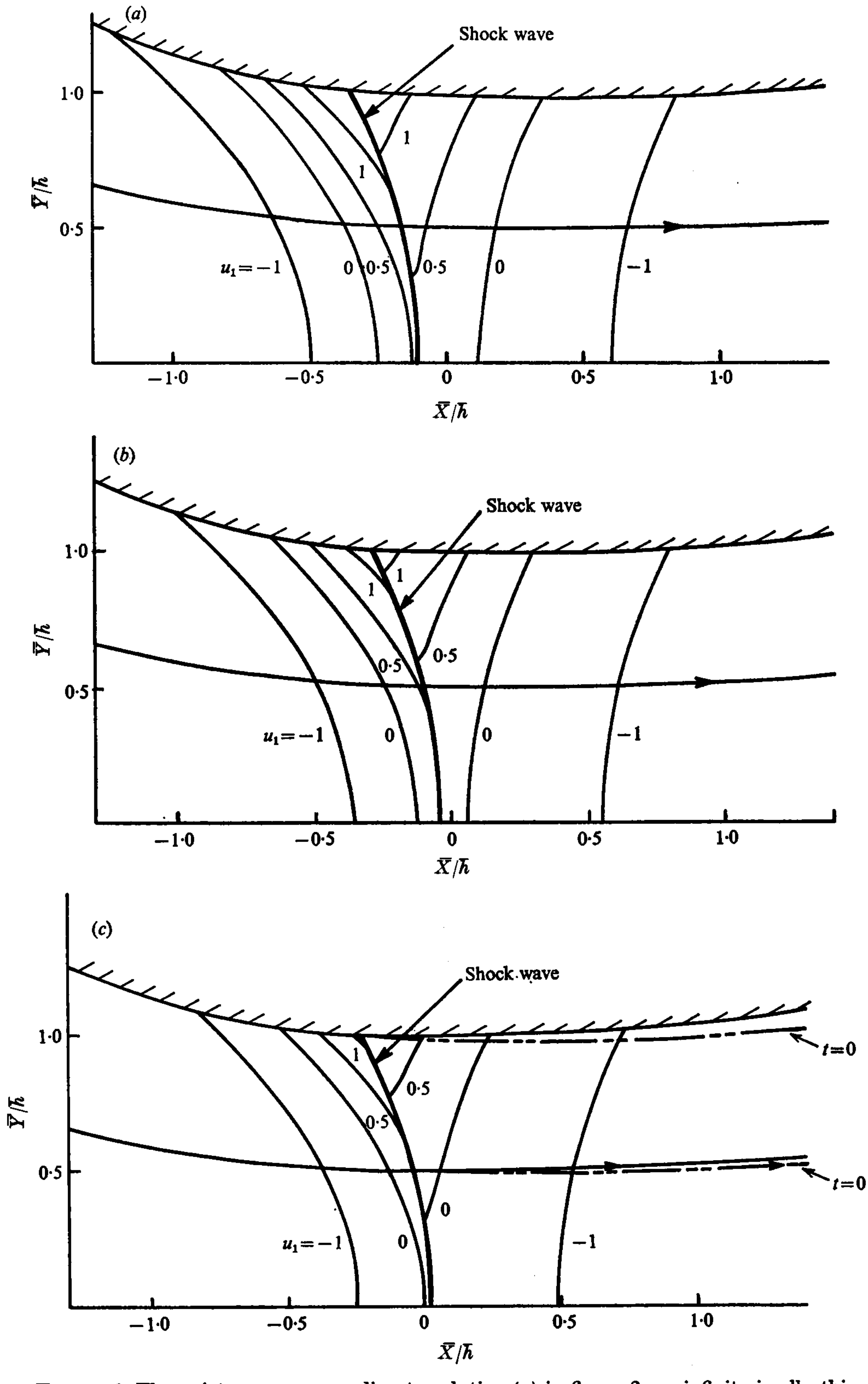
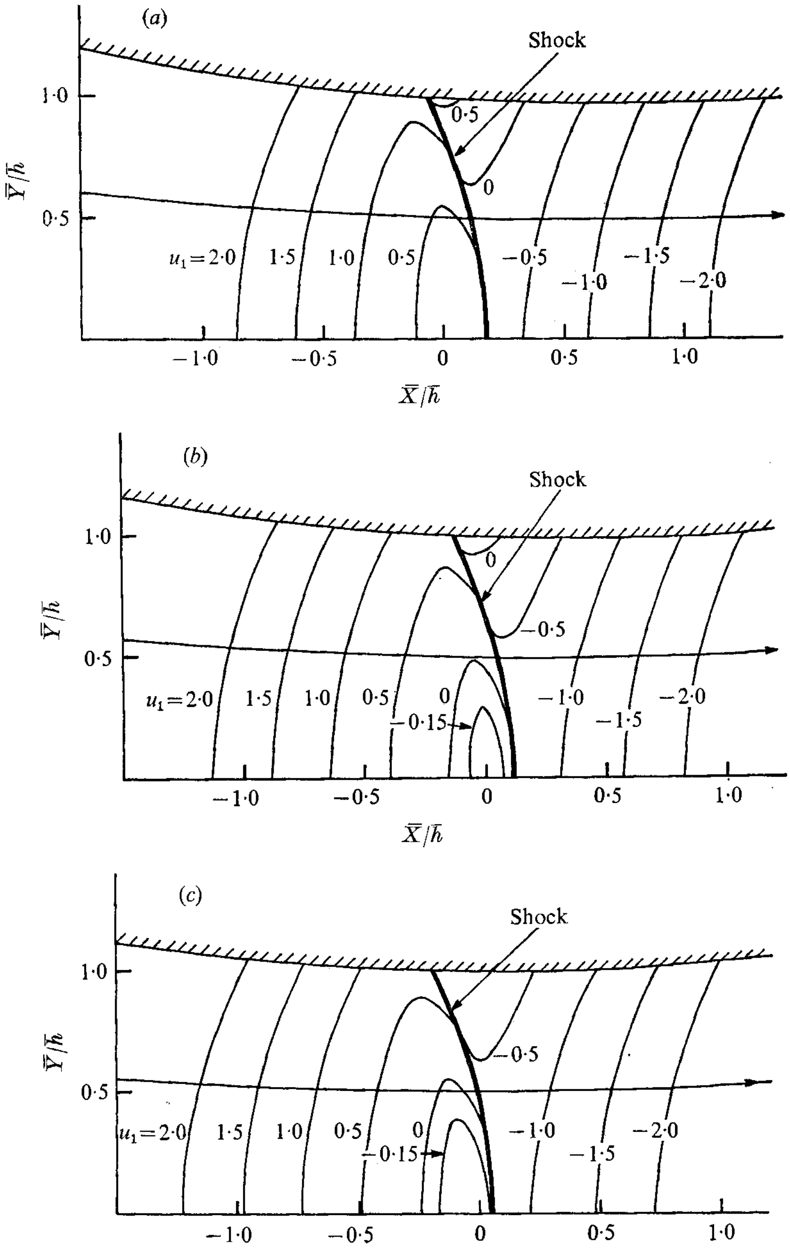


FIGURE 6. Flow pictures corresponding to solution (a) in figure 2, an infinitesimally thin shock imbedded in unsteady accelerating flow. $k_s = 0$, $\alpha_d = -0{\cdot}10$, $b = \frac{1}{2}$, $k = 1$, $\gamma = 1{\cdot}4$, $E_1 = 0{\cdot}1$, $\beta = \frac{1}{4}e^{-2t}$ (steady-state flow as $t \to \infty$, see (c)). —>—, streamline; ——, isotach. (a) $t = 0$, (b) $t = 0{\cdot}35$, (c) $t = \infty$, showing overall wall and streamline variation.



FIGURES 7 (*a*)–(*c*). For legend see p. 380.

In general, x_α and y_α can be functions of time; in the flow pictures shown here, they are calculated as follows. When k_s is not zero and a continuous $z(s)$ exists, x_α and y_α are the co-ordinates of the minimum of y_w (or y_{SL}) for the corresponding steady-state case ($\beta = 0$). Thus, x_α and y_α are constants and any wall motions are such that the wall is pinned at this point. When $k_s = 0$, the values of x_α and y_α

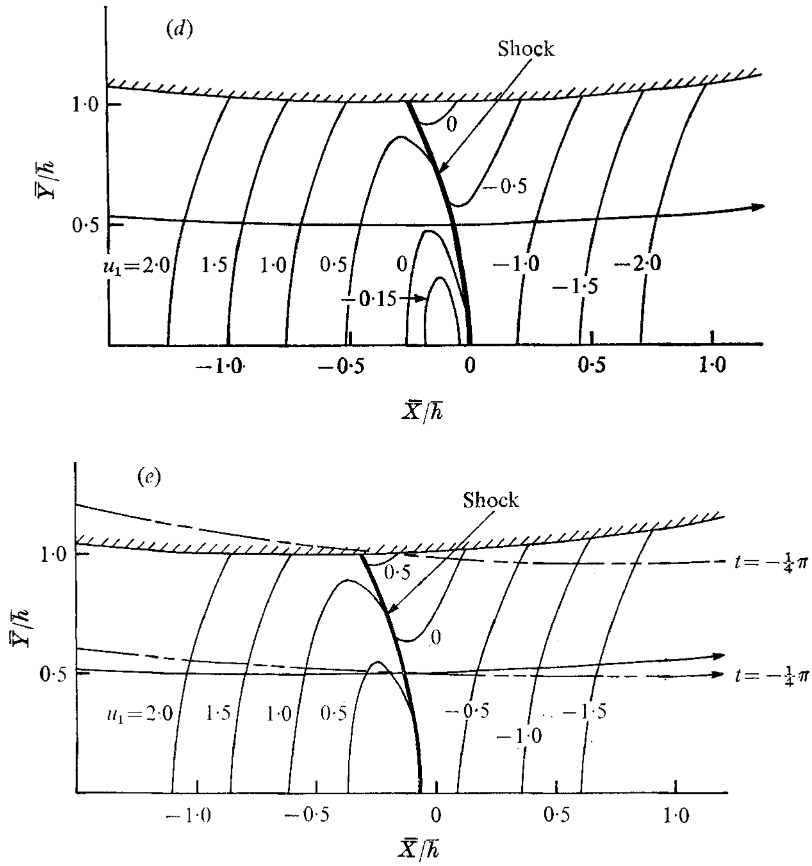


FIGURE 7. Flow pictures corresponding to solution (*b*) in figure 2, an infinitesimally thin shock imbedded in unsteady decelerating flow. $k_s = 0$, $\alpha_u = 0{\cdot}2$, $\alpha_d = -0{\cdot}4$, $b = \frac{1}{2}$, $k = 1$, $\gamma = 1{\cdot}4$, $E_1 = 0{\cdot}1$, $\beta = \frac{1}{4}\sin 2t$ (cyclic disturbance). —>—, streamline; ——, isotach. (*a*) $t = -\frac{1}{4}\pi$, (*b*) $t = -\frac{1}{12}\pi$, (*c*) $t = 0$, (*d*) $t = \frac{1}{12}\pi$, (*e*) $t = \frac{1}{4}\pi$, showing overall wall and streamline variation.

upstream of the shock are calculated in exactly the same way. Then the values of x and y at the intersection of the shock and wall (or streamline) are used for the calculations downstream of the shock, where z has jumped to a subsonic value. These latter x_α and y_α are thus time dependent.

Flow pictures illustrating the kinds of calculations which can be made are shown in figures 5–7. Figures 5(*a*), (*b*) and (*c*) show a sequence of solutions for $\beta = \frac{1}{4}e^{2-t}$, i.e. an exponentially decreasing disturbance, in a decelerating flow ($\alpha_u = 0{\cdot}50$, $\alpha_d = -0{\cdot}70$), for $k = 10^{-2}$. The $z(s)$ solution is similar to that shown in figure 3. Because $\beta \to 0$ as $t \to \infty$, figure 5(*c*) is the steady-state flow picture for the given parameters. Although the shock is quite thin, the structure, and in fact temporal variations in structure and position, can be seen quite clearly.

Figures 6(*a*), (*b*) and (*c*) show a sequence of flow pictures, again for $\beta = \frac{1}{4}e^{-2t}$ but now for an acclerating flow ($\alpha_d = -0{\cdot}10$), with $k_s = 0$; the $z(s)$ solution is curve (*a*) in figure 2. In this case, the shock is a discontinuity imbedded in the flow. In figure 6 (*a*), it is seen that the flow immediately downstream of the normal part of the wave is supersonic. This apparently anomalous behaviour is explained by recalling that the shock is moving also and that, relative to the shock, the proper subsonic flow exists in this region. This is borne out by consideration of figure 6 (*c*), the steady-state solution for the given parameters.

Finally, figures 7 (*a*), (*b*), (*c*), (*d*) and (*e*) show a sequence of flow pictures for $\beta = \frac{1}{4}\sin^{2t}$ (cyclic disturbance) in a decelerating flow ($a_u - 0{\cdot}2$, $\alpha_d = -0{\cdot}4$), for $k_s = 0$. The $z(s)$ solution is curve (*b*) in figure 2. A most interesting feature of this flow is the occurrence of a subsonic pocket in the supersonic flow ahead of the shock (figures 7*b*, *c*, *d*).

In figures 5 (*c*), 6 (*c*) and 7 (*e*), the overall variation of the walls and streamlines is shown. That is, as was mentioned previously, the boundary conditions are affected by the choice of β, in this similarity solution. This motion can be emphasized or minimized by the choice of parameters in β. All the flow pictures show solutions for the upper half-plane of a symmetrical channel flow; since for the time regime studied the wall is instantaneously a streamline, the flow through an asymmetric channel corresponds to the flow between the wall and the streamline shown. Finally, it should be mentioned that several other flow examples, including results for other values of k_s, are shown in Adamson (1972*b*).

6. Discussion

The general features of unsteady transonic channel flow with shock waves can be studied relatively simply with the present similarity solutions. Thus, the variation of the shock position and structure as well as the behaviour of the flow upstream and downstream of the shock can be analysed as a function of time for various flow disturbances. In addition, the effects of the Reynolds number on structure can be studied for steady and unsteady flows. Finally, both decelerating and accelerating channel flows can be considered. On the other hand, these solutions suffer from the fact that only special wall shapes can be considered and that the walls move with time. Furthermore, because the wall is instantaneously a streamline, it goes through a small change in slope where the wall and oblique shock intersect; this effect is too small to be seen to the scale of the drawings, but it exists, nevertheless. As a result of this change in channel shape, there is a small change in pressure distribution downstream of the shock.

It is clear from the above remarks that, in so far as direct applications are concerned, attention should be concentrated upon consideration of flows where the wall shape is specified. Such solutions evidently cannot be given by similarity solutions. Nevertheless, the similarity solutions shown here are extremely useful as a means of illustrating fundamental flow features in unsteady transonic flow.

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